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Propagating Fronts in a Bistable Coupled Map Lattice

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We consider the (traveling-wave-like) fronts which propagate with rational velocity p/q in a simple coupled map lattice for which the local map has two stable fixed points. We prove the uniqueness of such orbits up to time iterations, space translations, and permutations of the associated codes. A condition for their existence is also given, but it has to be checked in each case. We expect this condition to serve as a selection mechanism. The technique employed, the so-called (generalized) transfer matrix method, allows us to give explicit expressions for these solutions. These fronts are actually the observed orbits in the numerical simulations, as is shown with two examples: the case of velocity 1/2 and that of velocity 1.

KEY WORDS: Coupled map lattices; fronts; transfer matrices.

The appearance of fronts is known to be a relevant feature of real patterns in extended dynamical systems.[1] Generally, the fronts are created by the spatial juxtaposition of different type of *ideal* (i.e., homogeneous) solutions for which the time evolution is regular, namely the domains. Such structures are observed in reaction-diffusion chemical systems,[2] in alloy solidification,[3] and, with a more complex shape, in crystal growth surface by molecular beam epitaxy.[4] Though the dynamics of these coherent structures in space-time continuous models (i.e., the partial differential equations) is now well understood from a rigorous mathematical point of view,[5] the study is not as complete in discrete models.

On the other hand, coupled map lattices (CML) have been proposed as the simplest models of extended dynamical systems. These are space-time discrete dynamical systems with a continuous state representing the reaction-diffusion systems (see ref. 6 for a review).

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The phenomenology of the front dynamics in a one-dimensional bistable CML is the following.[7] Generically (i.e., if the local map is not symmetric), there is a bifurcation from a regime of standing fronts to a propagating front behavior at a (strictly) positive critical value of the coupling parameter. The existence of this positive critical value is due to the discreteness of space and is well known as the pinning effect in condensed matter physics.[9]

In a previous work,[8] the existence and the stability of the steady fronts was proven in a simple bistable CML. The method was based mainly on the use of area-preserving linear spatial maps, the so-called transfer matrices. Generalizing this idea, we prove here the uniqueness up to time iterations, space translations, and permutations of the associated code of the (traveling wave) fronts, and we write down explicit expressions for these orbits. The paper is organized as follows. First, the model, the patterns under consideration, and their associated temporal codes are defined. We also study some simple properties of the traveling interfaces and show how the use of the Banach fixed-point theorem may give an insight into the corresponding fronts. The construction of these configurations and the condition for their existence are detailed in the second section. Then the stability analysis is sketched and we discuss the selection of the temporal code from numerical results. Finally, two examples, the cases of velocity 1/2 and 1, are examined accurately in order to emphasize the correspondence with the numerical experiments.

1. DEFINITIONS

Let $M = [0, 1]^{\mathbb{Z}}$, endowed with the usual supremum norm, be the phase space of the CML under consideration for which the dynamics is given by the one-parameter family of maps:

$$F_\varepsilon:\quad M \to M$$
$$x^t \mapsto x^{t+1}$$

The model represents the simplest reaction-diffusion systems, i.e., the spatial interaction is the discrete Laplacian operator. Therefore the new state at time $t+1$ is given by[6]

$$x_i^{t+1} \equiv (F_\varepsilon x^t)_i = (1-\varepsilon)\, f(x_i^t) + \frac{\varepsilon}{2}\,(f(x_{i-1}^t) + f(x_{i+1}^t)) \qquad \forall i \in \mathbb{Z} \quad (1)$$

Here the diffusion coefficient $\varepsilon \in [0, 1]$ and the local map f is the simplest nonlinear bistable map of the interval which is not piecewise constant:

$$f(x) = \begin{cases} ax + (1-a)\, X_1 & \text{if } \ 0 \leqslant x < c \\ ax + (1-a)\, X_2 & \text{if } \ c \leqslant x \leqslant 1 \end{cases}$$

where the parameters a, X_1, X_2, and c obey the following inequalities:

$$0 < a < 1 \qquad \text{and} \qquad 0 \leqslant X_1 < c < X_2 \leqslant 1$$

which ensure the existence of the two stable fixed points X_1 and X_2, the only attractors for f. This map can be viewed as the simplest model of the autocatalytic reaction in chemical systems[2] or of the beam effect on the surface in crystal growth.[4] The patterns under consideration are now defined.

Definition 1.1. An interface is an orbit $\{x^t\}_{t \in \mathbb{N}}$ of the CML with the properties

$$\forall t, \exists j_t \in \mathbb{Z}: \quad \begin{cases} x_i^t < c & \forall i < j_t \\ x_i^t \geqslant c & \forall i \geqslant j_t \end{cases}$$

and for any given t,

$$\lim_{i \to -\infty} x_i^t = X_1, \qquad \lim_{i \to +\infty} x_i^t = X_2$$

The cases where $j_{t+1} = j_t \, \forall t$ are excluded here, since the attractor is then a steady interface, as was proven in ref. 8. Moreover, we know that a necessary and sufficient condition for the propagating interfaces (i.e., $j_{t+1} \geqslant j_t \, \forall t$) to appear in this model is

$$c > \frac{X_1 + X_2}{2} \qquad \text{and} \qquad \varepsilon > \varepsilon_c > 0 \tag{2}$$

Here the second inequality ensures the propagation for long times, while the first one gives the "direction" of propagation, to the "right" in this case.

Note that the following study is also valid for the anti-interface orbits by simply applying the symmetry $x_i \mapsto x_{-i-1}$ to any interface.

A first step in this study is to give a bound on the increments of j_t. Both from this local map and from the convex linear combination (1), we have the following result.

Proposition 1.2. $j_{t+1} \leqslant j_t + 1 \; \forall t \in \mathbb{N}$.

Among all the possible propagating structures, we are going to study the following ones:

Definition 1.3. Let p and q $(q \geqslant 1,\ p \leqslant q)$ be coprime numbers. A front is an interface such that

$$x^q \equiv F_\varepsilon^q x^0 = S^p x^0$$

where S stands for the (space) translation operator: $(Sx)_i = x_{i-1}$.

The sequence $X=(x^{(0)}, x^{(1)},..., x^{(q-1)})$ of elements in M is called the *generator* of the front if

$$x^l = x^{(l)}, \qquad l=1,..., q-1$$

Let $T_{p,q}$ be the set of velocity p/q fronts. The main goal of this paper is to determine the set $T_{p/q}$ and to compute explicitly the fronts. In order to consider disjoint sets of fronts, the orbits in $T_{p,q}$ must not coincide with any of their translated orbits except for the (mp)th ones $(m \in \mathbb{Z})$. Due to the fact that F_ε and S commute, it is easy to extend the first equality in (1.3) to any point of the orbit:

Proposition 1.4. A front satisfies the traveling wave equation

$$F_\varepsilon^q x^t = S^p x^t \qquad \forall t \tag{3}$$

Hence this orbit is entirely characterized by its generator

$$x^{nq+l} = S^{np} x^{(l)} \qquad \forall n \in \mathbb{N}, \quad l=1,..., q-1$$

and $T_{p,q}$ is completely determined by all the possible generators X.

We now describe the front properties in terms of the properties of the associated sequences. To do this, we first introduce the following.

Definition 1.5. The (temporal) code associated with a traveling interface is the sequence Δ defined by

$$\Delta_t = j_t - j_{t-1} \qquad \forall t > 0$$

Notice that $\Delta \in \{0, 1\}^{\mathbb{Z}}$ and that knowledge of the sequence j or j_0 and the code Δ is equivalent. We will use the term temporal code for both the sequence j and Δ unless it is ambiguous. One has the following result.

Proposition 1.6. For a front, the sequence $\{j_t\}_{t \in \mathbb{N}}$ is such that

$$\forall t \quad j_{t+q} = j_t + p$$

This statement follows directly from the relation (3). As a consequence, Δ is periodic with period q and is such that each of its subsequences of length q contains p 1's.

Furthermore, we will often use a terminology which allows us to restrict the present study to a representative of a class of fronts:

Definition 1.7. Two temporal codes Δ and $\tilde{\Delta}$ are equivalent if there is a $t_0 \in \mathbb{N}$ such that $\Delta_t = \tilde{\Delta}_{t+t_0}$ $\forall t > 0$.

The traveling interfaces x^t and $\tilde{x}^t$ are said to be equivalent if their associated codes are equivalent. One can compute all the equivalent orbits of a front from its generator.

We now state precisely the correspondence between the temporal codes and the fronts.

Theorem 1.8. Given a sequence $\{j_t\}_{t \in \mathbb{N}}$, then either the corresponding front does not exist or it is unique.

Proof. Given p and q, we consider the dynamical system

$$x^{\tau+1} = S^{-p} F_\varepsilon^q x^\tau \qquad \forall \tau \geq 0 \tag{4}$$

Assume that the sequence $\{j_t\}_{t \in \mathbb{N}}$ with the property (1.6) is given. Then one knows explicitly the operator F_ε^q. Moreover, the traveling interfaces for the model (1) can be thought of as being elements of the convex sets

$$C(\theta^t) = \{x^t \in M: (x_i^t - c)(\theta_i^t - c) \geq 0\}$$

given the spatial sequence (*spatial code*) $\theta^t = \{\theta_i^t\}_{i \in \mathbb{Z}}$, where

$$\theta_i^t = \begin{cases} X_1 & \text{if } \; i < j_t \\ X_2 & \text{if } \; i \geq j_t \end{cases} \tag{5}$$

For any θ, $C(\theta)$ is a closed set, since we can write

$$C(\theta) = \bigcap_{i \in \mathbb{Z}} g_\theta^{-1}([0, r])$$

where $(g_\theta(x))_i = (x_i - c)(\theta_i - c)$ is continuous and r is some real number. Now the set

$$\mathscr{C} = \{x \in C(\theta): (S^{-p} F_\varepsilon^q)^\tau x \in C(\theta) \; \forall \tau > 0\} \subset M$$

is either empty or not. In the first case, there is no fixed point of (4) in $C(\theta)$. If $\mathscr{C}$ is nonempty, since $S^{-p} F_\varepsilon^q x - S^{-p} F_\varepsilon^q y$ reduces to its linear part in this set and since this linear operator is a contraction, then there exists a unique front with such a temporal code. ∎

From this theorem and the previous comments on the equivalence of fronts, we can immediately deduce $T_{p,q}$. Let $N(p, q)$ be the number of non-equivalent fronts, for which the generator is denoted $(x_n^{(0)}, x_n^{(1)}, \ldots, x_n^{(q-1)})$.

Corollary 1.9. $T_{p,q}$ is the set of the CML orbits for which the initial conditions are in

$$\bigcup_{n=1}^{N(p,q)} \bigcup_{k\in\mathbb{Z}} \bigcup_{l=0}^{q-1} \{S^k x_n^{(l)}\}$$

provided $x_n^{(l)}$ exists.

In practice it is not feasible to check whether or not $\mathscr{C}$ is empty. Further, this technique gives no information on the (space-time) shape of the solution (except for the two codes θ and Δ) nor on the possible restrictions for the existence of such solutions.

2. THE FRONTS

In order to obtain a more complete knowledge of the orbits under consideration, the traveling wave equation must be solved for all the nonequivalent fronts. We detail here the construction of such solutions.

Assume that the integers p and $q \geqslant 1$ and the sequence j are given. From now on, we choose $j_0 = 0$ for simplicity. Equation (3) is going to be solved for one configuration, say $x^{(0)}$. The decay rates in the front tails are first computed by the (generalized) transfer matrices. Then the construction is completed by considering the dynamics in the "center" of the structure. One has to carry out this procedure for all the nonequivalent orbits. However, we shall see that the decay rates are independent of the configuration under consideration. Only the "central" part of the dynamics changes from one configuration to a nonequivalent one.

In order to deal with simpler dynamical equations we define $y^{(k)} \in \mathbb{R}^{\mathbb{Z}}$, the deviation vectors from the (local map) fixed points:

$$y_i^{(k)} = \begin{cases} x_i^{(k)} - X_1 & \text{if} \quad i \geqslant j_k \\ x_i^{(k)} - X_2 & \text{if} \quad i \geqslant j_k \end{cases} \qquad k = 0,..., q-1$$

The corresponding dynamics for these vectors is deduced from the CML dynamics and the sequence j.

To solve the traveling wave equation, $y^{(q)}$ has to be expressed in terms of $y^{(0)}$. We obtain the relation

$$y_i^{(q)} = a^q \sum_{l=-q}^{q} m_{|l|}^q y_{i+l}^{(0)} + C_i^{p,q}$$

where

$$m_l^q = \sum_{p=0}^{E((q-l)/2)} \binom{l+2p}{p} \binom{q}{l+2p} (1-\varepsilon)^{q-(l+2p)} \left(\frac{\varepsilon}{2}\right)^{l+2p}$$

Here E stands for the floor function and $\binom{n}{m}$ for the binomial coefficients. The constants $C_i^{p,q}$ vanish for $i < -q$ and for $i \geqslant q$ and they differ "noncommutatively" for nonequivalent configurations. Moreover, the translation operator $\hat{S}$ clearly acts on the deviations in the same way as S acts on the original variables.

The (lower) linear part of (3) is solved by computing the sequences of vectors

$$Y_i = (y_i, y_{i+1}, ..., y_{i+2q-1})^T$$

of $\mathbb{R}^{2q}$, which are related by the linear map

$$Y_{i-q} = A_{p,q} Y_{i-q+1} \qquad \forall i < -q$$

where

$$A_{p,q} = \begin{pmatrix} -\dfrac{m_{q-1}}{m_q} & -\dfrac{m_{q-2}}{m_q} & \cdots & -\dfrac{(a^q m_p - 1)}{a^q m_q} & \cdots & -\dfrac{m_0}{m_q} & \cdots & -\dfrac{m_{q-1}}{m_q} & -1 \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & \ddots & & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}$$

is called the *(generalized) transfer matrix*. (The superscripts are dropped unless ambiguity results, and T denotes the transpose.) Naturally, the procedure is identical for $i \geqslant q$ and the corresponding transfer matrix is denoted $\tilde{A}_{p,\tilde{q}}$. Some relevant properties of these matrices are claimed in the following proposition. Let R be the $2q \times 2q$ matrix

$$R = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 \\ 0 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

with the property that $R^{-1}=R$. For $A_{p,q}$ we have the following result (similar properties hold for $\tilde{A}_{p,q}$).

Proposition 2.1. (i) $\det A_{p,q}=1$.

(ii) The characteristic polynomial $P_{A_{p,q}}$ factorizes as

$$P_{A_{p,q}}(\lambda)=\left(\frac{2\lambda}{a\varepsilon}\right)^q\left[a^q\left(1-\varepsilon+\frac{\varepsilon}{2}\left(\lambda+\frac{1}{\lambda}\right)\right)^q-\lambda^p\right] \tag{6}$$

(iii) Let $\lambda_{i,p,q}$ be the ith eigenvalue of $A_{p,q}$. Then $V_{i,p,q}$, the corresponding eigenvector, is written

$$V_{i,p,q}=v_{i,p,q}((\lambda_{i,p,q})^{2q-1},..., \lambda_{i,p,q}, 1)^T$$

where $v_{i,p,q}$ is some real number. As a consequence, the eigenvalues are nondegenerate.

(iv) $A_{p,q}=R\tilde{A}_{p,q}^{-1}R$, which implies $\tilde{A}_{p,q}(RV_{i,p,q})=(1/\lambda_{i,p,q})(RV_{i,p,q})$.

Proof. Properties (i) and (iii) are due to the peculiar entries of these generalized transfer matrices. The second property is due partly to the peculiar form of $A_{p,q}$ and also partly to the expressions for m_l. Finally, to prove the last statement, one has to identify both the Y_i iterations (for $i<-q$) and those corresponding to the indexes $i\geqslant q$. ∎

For $x^{(0)}$ to be a front, we claim that both the initial vectors for these linear dynamics must belong to the contracting eigenspace of the transfer matrices. Let n (resp. $\tilde{n}$) be the number of $A_{p,q}$ (resp. $\tilde{A}_{p,q}$) contracting eigenvalues; then from (ii) and (iv) one obtains $n+\tilde{n}=2q$.

Though the study of the real spectra of these transfer matrices can be achieved analytically, the complex eigenvalues have been numerically computed for different values of p and q. We obtain $n=q+p$ and $\tilde{n}=q-p$.

In all cases, given these constraints, the general expression for the deviations corresponding to a front is

$$y_i^{(0)}=\sum_{l=1}^{n}\alpha_l^0\lambda_{l,p,q}^{-(i+1)} \qquad \forall i\leqslant -1$$

where α_i^0 is a real constant if $\lambda_{i,p,q}$ is real and $\alpha_i^0=\bar{\alpha}_j^0$ if $\lambda_{i,p,q}=\bar{\lambda}_{j,p,q}$ (the bar denotes the complex conjugate). A similar expression is obtained for the sites $i\geqslant 0$.

The $2q$ constants α_i^0 and $\tilde{\alpha}_i^0$ are computed by solving the $2q$ remaining lines (i.e., $-q\leqslant i<q$) of the traveling wave equation. These remaining equations can be transformed into the actions of spatial affine maps for

each of which the linear part is still the transfer matrix. In this way, we prove that, since $A_{p,q}$ is nonsingular, the constants are always uniquely determined.

For this solution $y^{(0)}$ to correspond to the deviations of a CML orbit, one has to check that the front thus constructed really is in the correct convex set $C(\theta')$ at each time step. Then the following must hold:

$$x_{j_k}^{(k)} \geqslant c \qquad \text{and} \qquad x_{j_k}^{(k+1)} < c \qquad \text{iff} \quad j_{k+1} = j_k + 1 \tag{7}$$

(provided that $x_i^{(k)} < x_{jk-1}^{(k)}$ if $i < j_k - 1$ and $x_i^{(k)} > x_{j_k}^{(k)}$ if $i > j_k$). It is not feasible to check these conditions in the general case, but we shall see with the examples that they greatly reduce the range of parameters for which this orbit exists. When it exists, the unique front with velocity p/q and given code j is given by (recall that $j_0 = 0$)

$$x_i^{(k)} = \begin{cases} X_1 + \sum\limits_{l=1}^{n} \alpha_l^s \lambda_{l,p,q}^{-i+s-1+(k-s)p/q} & \forall i \leqslant s-1 \\ X_2 + \sum\limits_{l=1}^{\tilde{n}} \tilde{\alpha}_l^s \tilde{\lambda}_{l,p,q}^{i-s-(k-s)p/q} & \forall i \geqslant s \end{cases}$$

$$\forall k = q_s, \ldots, q_{s+1},\ s = 0, \ldots, p-1 \tag{8}$$

where $\forall k = q_s, \ldots, q_{s+1}$, $j_k = j_{q_s} \equiv s$. The q_s are such that $j_{q_s+1} = j_{q_s} + 1$. The α_l^{s+1} are deduced from the α_l^s using the deviation dynamics.

We now briefly sketch the stability analysis of the fronts. In particular, we investigate the stability of the fixed points of the dynamical system (4). Let $j_{x^{(k)},t}$ (resp. $j_{x,t}$) be the temporal code of a front (resp. the initial configuration x). Assume that

$$\exists t_0 < \infty: \quad \forall t \quad j_{x,t} = j_{F_\varepsilon^{t_0} x^{(k)},t}$$

Then, by generalizing Theorem 1.8, we have that x converges toward an iteration of $x^{(k)}$. This implies the asymptotic stability of the front (though we have not shown that the initial conditions treated here form a neighborhood of $x^{(k)}$, for some topology in $[0, 1]^{\mathbb{Z}}$). From the numerical point of view, the simulation displays the convergence toward these structures. The only property required for the initial conditions seems to be that

$$\exists j, j'(j' \geqslant j): \quad \begin{cases} x_i < c & \text{if} \quad i < j \\ x_i \geqslant c & \text{if} \quad i \geqslant j' \end{cases}$$

At this stage, one may wonder if all the orbits with nonequivalent codes really exist in the physical situation (i.e., $\varepsilon \in [0, 1]$). From the numerical simulations, it turns out that only one front configuration solution of the traveling wave equation is selected by the system: the one with the *most uniform code*. One can see that the corresponding sequence j is given by

$$j_t = j_0 + E\left(t\frac{p}{q} + \delta\right) \qquad \forall t \in \mathbb{N}$$

where δ depends on the initial condition shape. We expect that a convexity argument should be invoked to prove this. We finally note that this problem of code selection does not occur for the velocities $1/q$ and $(q-1)/q$, since there is only one nonequivalent configuration in these cases.

3. EXAMPLES

We close the study of propagating interfaces by solving completely the traveling wave equation for two examples: the case of velocity 1/2 and that of velocity 1. The first case is interesting as the simplest example of a nontrivial velocity for which one can write explicitly condition (7) and compare it to the numerical simulation. Velocity 1 is a special case since it is the maximum propagating velocity.

For $p=1$ and $q=2$, the set of fronts is

$$T_{1,2}(\varepsilon) = \bigcup_{k\in\mathbb{Z}} \{S^k x^{(0)}, S^k x^{(1)}\}$$

We will consider the solution $x^{(0)}$ for which $j_1 = j_0 = 0$ and consequently $j_2 = 1$. In this situation, we have $n=3$ with the peculiarity that $\lambda_{2,1,2} = \bar{\lambda}_{1,1,2}$ and $\tilde{n} = 1$. From now on, the extra subscripts 1, 2 and superscripts (0) are dropped. The affine system of four equations obtained for the deviations is

$$\begin{cases} m_2 y_{-4} + m_1 y_{-3} + m_0 y_{-2} + m_1 y_{-1} + m_2 y_0 + \dfrac{a\varepsilon^2}{4}\delta = y_{-3} \\[2ex] m_2 y_{-3} + m_1 y_{-2} + m_0 y_{-1} + m_1 y_0 + m_2 y_1 + \dfrac{\varepsilon}{2}\left[1 + a\left(1 - \dfrac{3\varepsilon}{2}\right)\right]\delta = y_{-2} \\[2ex] m_2 y_{-2} + m_1 y_{-1} + m_0 y_0 + m_1 y_1 + m_2 y_2 \\ \qquad + \left\{1 - \dfrac{\varepsilon}{2}\left[1 + a\left(1 - \dfrac{3\varepsilon}{2}\right)\right]\right\}\delta = y_{-1} \\[2ex] m_2 y_{-1} + m_1 y_0 + m_0 y_1 + m_1 y_2 + m_2 y_3 - \dfrac{a\varepsilon^2}{4}\delta = y_0 \end{cases}$$

where $\delta = X_2 - X_1$, $m_0 = a^2[(1-\varepsilon)^2 + \varepsilon^2/2]$, $m_1 = a^2(1-\varepsilon)\,\varepsilon$, and $m_2 = (a\varepsilon/2)^2$. The resulting constants α_l and $\tilde{\alpha}_l$ are

$$\begin{cases} \alpha_1 = \dfrac{\lambda_1^2[-R(1-\bar{\lambda}_1)(1-\lambda_3)(1-\tilde{\lambda}_1) - S\bar{\lambda}_1\lambda_3 + T\tilde{\lambda}_1]}{(\lambda_1-\bar{\lambda}_1)(\lambda_1-\lambda_3)(1-\lambda_1\tilde{\lambda}_1)} \\ \alpha_2 = \bar{\alpha}_1 \\ \alpha_3 = \dfrac{\lambda_3^2[-R(1-\lambda_1)(1-\bar{\lambda}_1)(1-\tilde{\lambda}_1) - S\lambda_1\bar{\lambda}_1 + T\tilde{\lambda}_1]}{(\lambda_1-\lambda_3)(\bar{\lambda}_1-\lambda_3)(1-\lambda_3\tilde{\lambda}_1)} \\ \tilde{\alpha}_1 = \dfrac{\tilde{\lambda}_1^2[R(1-\lambda_1)(1-\bar{\lambda}_1)(1-\lambda_3) - S\lambda_1\bar{\lambda}_1\lambda_3 + T]}{(1-\lambda_1\tilde{\lambda}_1)(1-\bar{\lambda}_1\tilde{\lambda}_1)(1-\lambda_3\tilde{\lambda}_1)} \end{cases}$$

where $R = \delta/a$, $S = 2(1-a)\,\delta/a^2\varepsilon$, and $T = (1/a + \varepsilon/2)(1-a)\,\delta/m_2$. Here the lattice values that enter in the existence condition are

$$\begin{cases} x_0^{(1)} = X_2 + \tilde{\alpha}_1 \tilde{\lambda}_1^{-1/2} \\ x_0^{(2)} = x_{-1}^{(0)} = X_1 + \alpha_1 + \bar{\alpha}_1 + \alpha_3 \end{cases}$$

By computing numerically the contracting eigenvalues, one can compare the range of ε values for which this front exists (Fig. 1) with the one obtained from direct numerical simulations of the CML (Fig. 2). From these pictures it is clear that the two values of ε bounding this range are

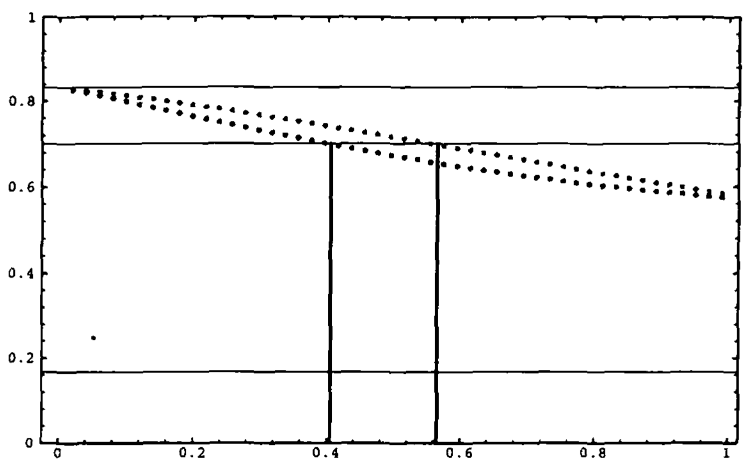

Fig. 1. $x_{-1}^{(0)}$ (lower dots) and $x_0^{(1)}$ (upper dots) versus ε for $a = 0.4$, $X_1 = 1/6$, $X_2 = 5/6$ and $c = 0.7$. The horizontal lines show the values at X_1, c and X_2 and the vertical lines show the range for the velocity 1/2 front's existence.

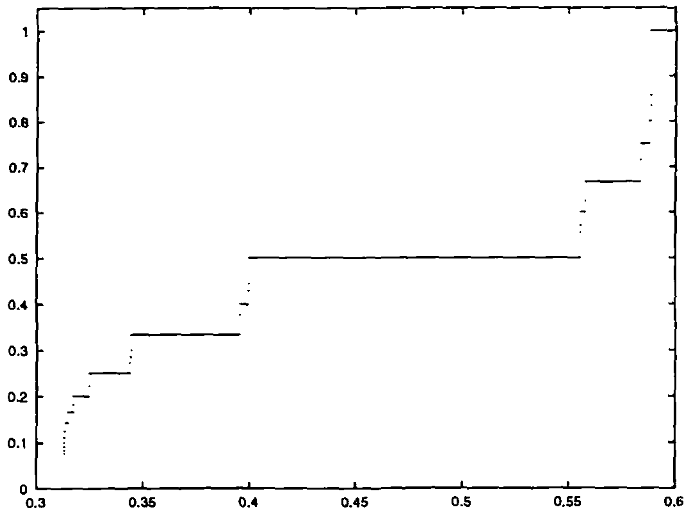

Fig. 2. Plot of the front's velocities versus ε measured from direct numerical simulations of the CML ($a = 0.4$, $X_1 = 1/6$, $X_2 = 5/6$ and $c = 0.7$). The critical value ε_c at which the front begins to propagate is 0.3129.... The value $\varepsilon_{-,1}$ at which the front with velocity 1 appears is 0.5882... (see text).

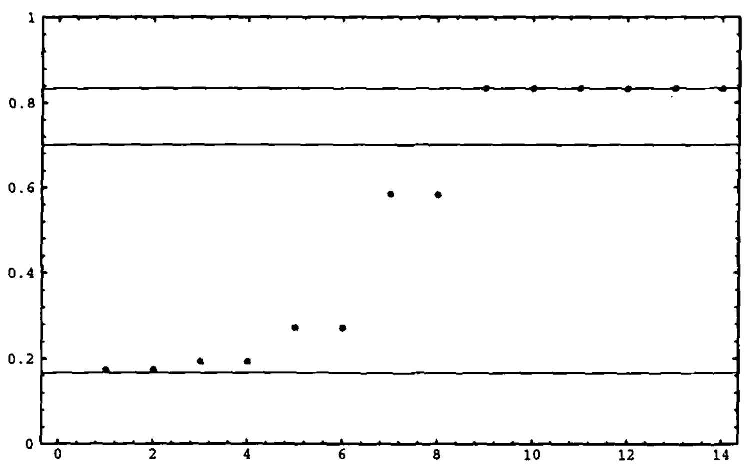

Fig. 3. The velocity 1's front configuration $x^{(0)}$ for $\varepsilon = 1$ ($a = 0.4$, $X_1 = 1/6$, $X_2 = 5/6$ and $c = 0.7$).

in good agreement with the "experimental" data. The devil's staircase structure of the velocity might be related to the unimodular transformations.[10]

The velocity 1 situation is slightly different, since the transfer matrices are not of determinant unity. The result of the spectral study of these two matrices is quite surprising. Indeed, for $0<a<1$ and $0<\varepsilon\leqslant 1$, $A_{1,1}$ has two contracting eigenvalues λ_1 and λ_2, but $\tilde{A}_{1,1}$ has no contracting eigenvalue. Then the front with velocity 1 is (assume again that $j_0=0$)

$$x_i^k=\begin{cases} X_1+\alpha_1\lambda_1^{-(i-k+1)}+\alpha_2\lambda_2^{-(i-k+1)} & \text{if } \; i<k \\ X_2 & \text{if } \; i\geqslant k \end{cases} \qquad \forall k \tag{9}$$

where the constants α_1 and α_2 are the solutions of the following system of equations:

$$\begin{cases} \alpha_1+\alpha_2=\dfrac{1-\varepsilon/2}{1-a\varepsilon/2}(X_2-X_1) \\[2ex] \alpha_1-\alpha_2=\dfrac{\varepsilon+a(1-3\varepsilon/2)}{(1-a\varepsilon/2)\sqrt{a[a+2\varepsilon(1-a)]}}(X_2-X_1) \end{cases}$$

Conditions (7) for the existence of this solution here become

$$X_2>c \qquad \text{and} \qquad X_1+\alpha_1+\alpha_2<c$$

The first condition is always satisfied, so there is no upper bound for the existence of the velocity 1 solution. Moreover, one can check that the second condition is fulfilled when

$$\varepsilon>\varepsilon_{-,1}\equiv\frac{2(X_2-c)}{X_2-X_1-a(c-X_1)}$$

For the values of the parameters given in Fig. 1, one obtains $\varepsilon_{-1}\simeq 0.59$ (see Fig. 2). A representation of this solution for the limiting case $\varepsilon=1$ is plotted in Fig. 3. Note the somewhat unexpected result obtained by choosing $\varepsilon=1$ in (9):

$$x_{2i-k}^k=x_{2i+1-k}^k \qquad \forall i\leqslant k-1$$

4. CONCLUSION

In this paper, we have presented a method for computing explicitly the fronts which propagate with a rational velocity in a (simple) bistable CML. Among these orbits, those that have the most uniform temporal code are the ones that appear in numerical simulations of the physical situation. However, to establish the equality of the numerical results with our analytical solutions, the condition for the fronts' existence must be checked separately for each code and the stability analysis must be done more carefully. We believe that the configurations with nonuniform codes do not appear. Moreover, we have not proved the velocity's increase with the coupling strength, which can be presumed from Fig. 2, nor that, given ε, there exists a unique possible generator for the front orbits.

From the present study and by generalizing the results of ref. 8, one can see that the phenomenon of front propagation in a bistable CML results from a succession of generalized saddle-node bifurcations [of the frontlike fixed points of the mappings (4)], though it is not clear that for each stable front, there corresponds an unstable one. At each bifurcation point, one front disappears while a new one appears (with a new velocity).

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